Practice Assessment 17

Differentiation and Integration of Power Series

Term-by-Term Differentiation of Power Series: Suppose that the power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ converges on the interval $(a-R, a+R)$ for some $R>0$, and the function f is defined to be its sum:

$$f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k, \quad x \in (a-R, a+R).$$

Then the function f is differentiable for $x \in (a-R, a+R)$ and we can find f' by differentiating the series term-by-term:

$$f'(x) = \sum_{k=1}^{\infty} kc_k(x-a)^{k-1}, \quad x \in (a-R, a+R).$$

The resulting series converges absolutely for $x \in (a-R, a+R)$.

1. Find a power series representation for the given function by differentiating a known power series. State the radius of convergence and interval of convergence.

(a) $f(x) = \dfrac{2}{(1+x)^2}$ $\qquad \left(Hint: f(x) = -2\dfrac{d}{dx}\left(\dfrac{1}{1+x}\right)\right)$

(b) $f(x) = \dfrac{x}{(1+x^2)^2}$

(c) $f(x) = \dfrac{1-x^2}{(1+x^2)^2}$ $\qquad \left(Hint: f(x) = \dfrac{d}{dx}\left(\dfrac{x}{1+x^2}\right)\right)$

Term-by-Term Integration of Power Series: Suppose that the power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ converges on the interval $(a-R, a+R)$ for some $R > 0$, and the function f is defined to be its sum:

$$f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k, \quad x \in (a-R, a+R).$$

Then the function f is integrable and we can find $\int f(x)\,dx$ by integrating the series term-by-term:

$$\int f(x)\,dx = \sum_{k=1}^{\infty} c_k \frac{(x-a)^{k+1}}{k+1}, \quad x \in (a-R, a+R).$$

The resulting series converges absolutely for $x \in (a-R, a+R)$.

2. Find a power series representation for the given function by integrating a known power series. State the radius of convergence and interval of convergence.

(a) $f(x) = \ln(1+x)$

(b) $f(x) = \ln(1-x)$

(c) $f(x) = x\ln(1+x)$

(d) $f(x) = \arctan(2x)$