AMAT112: Calculus I

Limits

1. If the functions behave "nicely" finding their limits can seem indistinguishable from just evaluating the functions at that point - just remember **taking a limit and plugging in a value are NOT the same thing**. For example, each of the following limits have the same answer as simply "plugging in the point".

   (a) $\lim_{x \to -3} x^2 - 13$

   (b) $\lim_{x \to 2} \dfrac{2x+5}{11-x^3}$

   (c) $\lim_{y \to -3} (5-y)^{4/3}$

   (d) $\lim_{y \to 2} \dfrac{y+2}{y^2+5y+6}$

   (e) $\lim_{z \to 4} \sqrt{z^2-10}$

   (f) $\lim_{h \to 0} \dfrac{3}{\sqrt{3h+1}+1}$

2. But finding limits isn't always as easy as those above. These examples consist of functions that can be evaluated at the point, but the limits are not simply "plugging in the point".

   (a) $\lim_{x \to 2} f(x)$, if $f(x) = \begin{cases} 3x+7 & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$

   (b) $\lim_{t \to -3} v(t)$, if $v(t) = \begin{cases} t^2-1 & \text{if } t > -3 \\ 12+x & \text{if } t \leq -3 \end{cases}$

3. Piecewise functions, like those in the previous problem, are great examples of functions where we need to consider limits from both the left and the right. In the following examples calculate both left and right sided limits and determine whether the limit exists.

   (a) $\lim_{x \to 1} f(x)$ if $f(x) = \begin{cases} 1-x^2 & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

   (b) $\lim_{x \to -2} (x+3)\dfrac{|x+2|}{x+2}$

   (c) $\lim_{y \to 1} \dfrac{\sqrt{2y}\,(y-1)}{|y-1|}$

   (d) $\lim_{x \to -2} f(x)$ if $f(x) = \begin{cases} x^2+5 & \text{if } x < -2 \\ 1-3x & \text{if } x \geq -2 \end{cases}$

   (e) $\lim_{t \to 1} \dfrac{t^2-1}{|t-1|}$

   (f) $\lim_{u \to -2} |u+2| + u^2$

4. Then there are instances where we cannot even evaluate the function at the point we want to find the limit for.

   (a) These limits can be found by first simplifying the expressions by either combining or factoring.

   i. $\lim_{x \to 5} \dfrac{x-5}{x^2-25}$

   ii. $\lim_{t \to -5} \dfrac{t^2+3t-10}{t+5}$

   iii. $\lim_{y \to 1} \dfrac{y^2+y-2}{y^2-1}$

   iv. $\lim_{z \to -2} \dfrac{-2z-4}{z^3+2z^2}$

   v. $\lim_{x \to -3} \dfrac{x+3}{x^2+4x+3}$

   vi. $\lim_{a \to 2} \dfrac{a^2-7a+10}{a-2}$

   vii. $\lim_{x \to 1} \dfrac{1}{x-1} - \dfrac{2}{x^2-1}$

   viii. $\lim_{h \to 0} \dfrac{(x+h)^2-x^2}{h}$

   ix. $\lim_{h \to 0} \dfrac{(x+h)^3-x^3}{h}$

   It's important to remember that we can cancel out the terms that prevent us from "plugging in the point" since we are never actually evaluating the function at the limit point.

   (b) These limits can be found by multiplying by the conjugate.

   i. $\lim_{x \to 1} \dfrac{x-1}{\sqrt{x+3}-2}$.

   ii. $\lim_{t \to 2} \dfrac{\sqrt{t^2+12}-4}{t-2}$.

   iii. $\lim_{u \to -3} \dfrac{2-\sqrt{u^2-5}}{u+3}$.

   iv. $\lim_{y \to -1} \dfrac{\sqrt{y^2+8}-3}{y+1}$.

   v. $\lim_{s \to -2} \dfrac{s+2}{\sqrt{s^2+5}-3}$.

   vi. $\lim_{h \to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$.

   (c) These limits cannot be manipulated algebraically to make them simpler. We can employ the **Sandwich Theorem** if we recognise that a function behaves like functions we are more familiar with around the point we're interested in.

   i. $\lim_{x \to 0} f(x)$, if $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$, for $-1 \le x \le 1$.

   ii. $\lim_{t \to 0} g(t)$, if $2-t^2 \le g(t) \le 2\cos(t)$, for all t.

   iii. $\lim_{y \to 0} \dfrac{y\sin(y)}{2-2\cos(y)}$, if $1-\dfrac{y^2}{6} < \dfrac{y\sin(y)}{2-2\cos(y)} < 1$, for y near 0.

   iv. $\lim_{\theta \to 0} \dfrac{1-\cos(\theta)}{\theta^2}$, if $\dfrac{1}{2}-\dfrac{\theta^2}{24} < \dfrac{1-\cos(\theta)}{\theta^2} < \dfrac{1}{2}$, for small θ.

   Sometimes you'll know what the function you're interested looks like and sometimes you won't. Applying the Sandwich Theorem can make a problem quite straight forward, so work on identifying what kind of problems call for its use. In general, you want to find a way of bounding any factor or term that doesn't behave well.

(d) Some limits can be found by using knowledge of other limits and some algebraic manipulation. For example, each of the following limits can be found by using the fact that $\lim\limits_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$.

i. $\lim\limits_{t \to 0} \frac{\sin(t)}{12t}$

ii. $\lim\limits_{x \to 0} \frac{\sin(-8x)}{2x}$

iii. $\lim\limits_{x \to 0} \frac{\tan(x)}{x}$

iv. $\lim\limits_{z \to 0} \frac{\tan(5z)}{\sin(3z)}$

v. $\lim\limits_{x \to 2} \frac{\sin(x^2 - 6x + 8)}{x^2 - 6x + 8}$

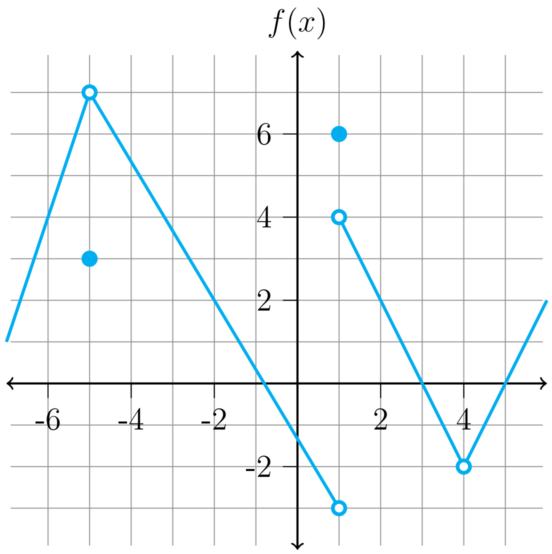
vi. $\lim\limits_{x \to 0} \frac{\sin(\sin(x))}{\sin(x)}$

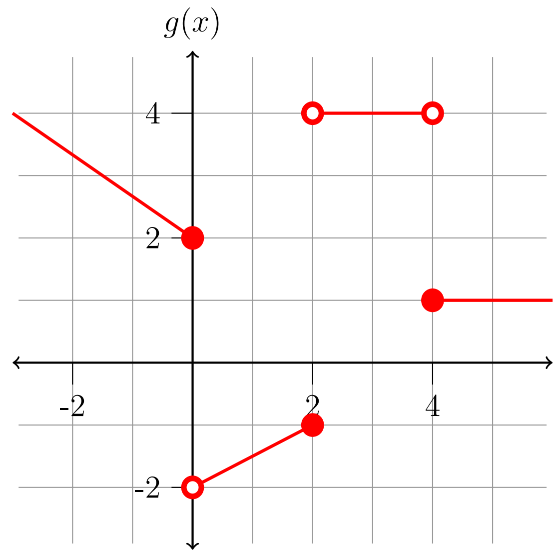
vii. $\lim\limits_{x \to 0} \frac{1 - \cos(x)}{x^2}$

viii. $\lim\limits_{x \to 0} \frac{x \sin(x)}{1 - \cos(x)}$

ix. $\lim\limits_{t \to 0} \frac{\sqrt{1 - \cos(t)}}{t}$

5. Finally, we can calculate limits of a function if we are given its graph, even if we don't have an algebraic expression for it. For the functions graphed below, find the indicated limits. (These problems were taken from Paul's Online Notes)





(a) i. $\lim\limits_{x \to -5} f(x)$ ii. $\lim\limits_{x \to -2} f(x)$ iii. $\lim\limits_{x \to 1} f(x)$ iv. $\lim\limits_{x \to 4} f(x)$

(b) i. $\lim\limits_{x \to -1.5} g(x)$ ii. $\lim\limits_{x \to 0} g(x)$ iii. $\lim\limits_{x \to 2} g(x)$ iv. $\lim\limits_{x \to 4} g(x)$

(c) i. $\lim\limits_{x \to -1} f(g(x))$ ii. $\lim\limits_{x \to 0} f(g(x))$ iii. $\lim\limits_{x \to 1} g(f(x))$ iv. $\lim\limits_{x \to 5} f(g(x))$

Answers

1. (a) -4
   (b) 3
   (c) 16
   (d) $\frac{1}{5}$
   (e) $\sqrt{6}$
   (f) $\frac{3}{2}$

2. (a) 13
   (b) DNE

3. (a) 0
   (b) DNE
   (c) DNE
   (d) DNE
   (e) DNE
   (f) 4

4. (a)
   i. $\frac{1}{10}$
   ii. -7
   iii. $\frac{3}{2}$
   iv. $-\frac{1}{2}$
   v. $-\frac{1}{2}$
   vi. -3
   vii. $\frac{1}{2}$
   viii. $2x$
   ix. $3x^2$

   (b)
   i. 4
   ii. $\frac{1}{2}$
   iii. $\frac{3}{2}$
   iv. $-\frac{1}{3}$
   v. $-\frac{3}{2}$
   vi. $\frac{1}{2\sqrt{x}}$

   (c)
   i. $\sqrt{5}$
   ii. 2
   iii. 1
   iv. $\frac{1}{2}$

   (d)
   i. $\frac{1}{12}$
   ii. -4
   iii. 1
   iv. $\frac{5}{3}$
   v. 1
   vi. 1
   vii. $\frac{1}{2}$
   viii. 2
   ix. DNE.

5. (a)
   i. 7
   ii. 2
   iii. DNE
   iv. -2

   (b)
   i. 3
   ii. DNE
   iii. DNE
   iv. DNE

   (c)
   i. 0
   ii. 2
   iii. 4
   iv. 6