AMAT112: Calculus I

Limits at Infinity

1. Calculate the following limits.

a) $\lim\limits_{x\to\infty} \sqrt{\dfrac{8x^2-3}{2x^2+x}}$

b) $\lim\limits_{t\to-\infty} \left(\dfrac{t^2+t-1}{8t^2-3}\right)^{1/3}$

c) $\lim\limits_{y\to-5} \dfrac{2y}{2y+10}$

d) $\lim\limits_{x\to\infty} \dfrac{\sin^2(x)}{x^2}$

e) $\lim\limits_{u\to\infty} \dfrac{\sqrt{u^2+1}}{u+1}$

f) $\lim\limits_{v\to-\infty} \dfrac{\sqrt[3]{v}-5v+3}{2v+v^{2/3}-4}$

g) $\lim\limits_{x\to\infty} \dfrac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}}$

h) $\lim\limits_{s\to7} \dfrac{4}{(s-7)^2}$

i) $\lim\limits_{t\to0} \dfrac{2}{t^{1/5}}$

j) $\lim\limits_{s\to-\infty} \dfrac{\sqrt[3]{s}-\sqrt[5]{s}}{\sqrt[3]{s}+\sqrt[5]{s}}$

k) $\lim\limits_{a\to0} \dfrac{-1}{a^2(a+1)}$

l) $\lim\limits_{t\to\infty} \dfrac{\cos^6(t^5+7t+1)}{t^2+3}$

m) $\lim\limits_{x\to0} \dfrac{x^2-3x+2}{x^3-2x^2}$

n) $\lim\limits_{t\to0} \dfrac{\sqrt{t^6+3t+1}}{t^3+t^2}$

o) $\lim\limits_{y\to0} \dfrac{y^{17}+y^{11}-5y^2}{y^{16}-y^5+2y^2}$

p) $\lim\limits_{z\to0} \dfrac{\sqrt[4]{z^{24}+z^{16}-z^{15}+1}}{\sqrt[3]{z^{18}-32z^2-1}}$

2. Recall we have seen two types of asymptotes so far: vertical and horizontal. The function $y=f(x)$ has a *vertical asymptote* at $x=a$ if at least one of the limits $\lim\limits_{x\to a^-} f(x)$ or $\lim\limits_{x\to a^+} f(x)$ is ∞ or $-\infty$. The function $y=f(x)$ has a *horizontal* asymptote at $y=c$ if at least one of $\lim\limits_{x\to-\infty} f(x)$ or $\lim\limits_{x\to\infty} f(x)$ is equal to c.

For the following functions, find all vertical and horizontal asymptotes.

a) $f(x) = \dfrac{4x^3+8x^2-11x+3}{x^3+2x^2-x-2}$

b) $v(t) = \dfrac{3t^3-21t-18}{2t^3+2t^2-42t-90}$

c) $g(z) = \dfrac{-z^2+3z-2}{z^4-5z^2+4}$

d) $u(t) = \dfrac{\pi(t^2-1)(t^2-9)(t^2-16)}{et^2(t+1)(t+2)(t+3)(t+4)}$

3. Write the equation of a function $f(x)$ that satisfies each of the following conditions.

   a) $f(1) = 5, \quad \lim_{t \to 3} f(t) = \infty, \quad \lim_{t \to 6^-} f(t) = \infty, \quad \lim_{t \to 6^+} f(t) = -\infty, \quad \lim_{t \to \pm\infty} f(t) = -2$

   b) $f(0) = \pi, \quad \lim_{z \to -\infty} f(z) = -1, \quad \lim_{z \to \infty} f(z) = 1$

   c) $f(0) = 7, \quad \lim_{x \to 6} f(x) = \infty, \quad \lim_{x \to -\infty} f(x) = 5, \quad \lim_{x \to \infty} f(x) = 9$

   d) Write your birthday down in the format $d/m/2000 + y$.
   $f(0) = d, \quad \lim_{x \to \frac{m}{2}^-} f(x) = \infty, \quad \lim_{x \to \frac{m}{2}^+} = -\infty, \quad \lim_{x \to m} = -\infty, \quad \lim_{x \to \pm\infty} f(x) = 2y$

4. As the independent variable approaches $\pm\infty$, the behaviour of a function is often determined by only a few terms. We refer to these as *dominant* terms. For each of the following functions, determine which are the dominant terms when calculating the limits going to each of $-\infty$ and ∞ and calculate these limits.

   a) $f(x) = \dfrac{6e^{4x} - e^{-3x}}{12e^{4x} + e^{x} + e^{-3x}}$

   b) $s(t) = \dfrac{3e^{2t} - e^{-t}}{2e^{2t} + e^{-2t}}$

   c) $g(z) = \dfrac{e^{-z} + z^2 + 3}{z^2 + 2z + 2}$

   d) $h(y) = \dfrac{e^{y} + 12y^{1/3} + y^3}{e^{-y} + y^{1/3}}$

   e) $k(a) = \dfrac{10\sqrt[3]{e^{-a} + \sin(a)} + a^{2/5}}{\sqrt[10]{a^4 + a^2 + 1} + e^{-a/3}}$

   f) $\theta(t) = \dfrac{e^{-t^2} + 7t^3 + 6}{e^{-t} + 6t^3 - 5}$

5. Carefully calculate (algebraically) the following limits.

   a) $\lim_{x \to \infty} \sqrt{x+9} - \sqrt{x+4}$

   b) $\lim_{t \to -\infty} \sqrt{x^2 + 3} + x$

   c) $\lim_{y \to \infty} \sqrt{y^2 + 3y} - \sqrt{y^2 - 2y}$

   d) $\lim_{a \to -\infty} 2a + \sqrt{4a^2 + 3a - 2}$

Answers

1.
a) 2
b) $\dfrac{1}{2}$
c) DNE
d) 0
e) 1
f) $-\dfrac{5}{2}$
g) ∞
h) ∞
i) DNE
j) -1
k) $-\infty$
l) 0
m) $-\infty$
n) ∞
o) $-\dfrac{5}{2}$
p) -1

2.
a) Vertical: $x = -2, -1, 1$, Horizontal: $y = 4$
b) Vertical: $t = -3, 5$, Horizontal: $y = \dfrac{3}{2}$
c) Vertical: $z = -2, -1$, Horizontal: $y = 0$
d) Vertical: $t = -2, 0$, Horizontal: $y = \dfrac{\pi}{e}$

3.
a) $f(t) = \dfrac{-2t^3 - 98}{(t-3)^2\,(t-6)}$
b) $f\,(z) = \dfrac{e^x - x^{-x} + 2\pi}{e^x + e^{-x}}$
c) $f\,(x) = \dfrac{(x^2+36)\,(9e^x + 5e^{-x})}{(x-6)^2\,(e^x + e^{-x})}$
d) Answers will vary.

4.
a) $\displaystyle\lim_{x\to-\infty} f\,(x) = -1, \quad \lim_{x\to\infty} f\,(x) = \frac{1}{2}$
b) $\displaystyle\lim_{t\to-\infty} s\,(t) = 0, \quad \lim_{t\to\infty} s\,(t) = \frac{3}{2}$
c) $\displaystyle\lim_{z\to-\infty} g\,(z) = \infty, \quad \lim_{z\to\infty} g\,(z) = 1$
d) $\displaystyle\lim_{y\to-\infty} h\,(y) = 0, \quad \lim_{y\to\infty} h\,(y) = \infty$
e) $\displaystyle\lim_{a\to-\infty} k\,(a) = 10, \quad \lim_{a\to\infty} k\,(a) = 1$
f) $\displaystyle\lim_{t\to-\infty} \theta\,(t) = 0, \quad \lim_{t\to\infty} \theta\,(t) = \frac{7}{6}$

5.
a) 0
b) 0
c) $\dfrac{5}{2}$
d) $-\dfrac{3}{4}$